

A Theorem of Van Kampen Type for Pseudo Peano Continuum Spaces

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Abstract

G. Conner and K. Eda (Topology and its Applications, 146, (2005), 317-328.) introduced a new construction of spaces from groups. They remarked that the construction is not categorical. In this paper, based on the work of Conner and Eda, we construct two new categories for which the functor of the fundamental group π has a right adjoint and consequently is right exact and preseves direct limits. Also, we study the behavior of the functor π on quotient spaces and give a new version of Van Kampen theorem for join of spaces in the new categories.

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1 Introduction and Motivation

K. Eda [2] proved that if X and Y are one-dimensional, locally path connected, path connected, metric spaces which are not semilocally simply connected at any point and have isomorphic fundamental groups, then X and Y are homeomorphic.

G. Conner and K. Eda [1] based on the above fact proposed a new construction of a space from a group and showed that this construction arises from the fundamental group of a space of the above type and it is homeomorphic to the topological space itself. They also remarked that the construction of topological spaces from groups is not categorical [1, Remark 2.8 (2)]. They called a point $x \in X$ a *wild point*, if X is not semilocally simply connected at x and denoted X^w for the subspace consisting of all wild points of X . They also called a space X *wild* if $X = X^w$.

In this paper, we call, for abbreviation, a one-dimensional, locally path connected, path connected, metric space as a *pseudo Peano continuum* and study some interesting properties of these spaces in section three in order to find some functorial properties related to the Conner-Eda constructions. In section four, we construct two new categories \mathbf{sqTop}^w and $\pi(\mathbf{Top}^w)$ for which the functor $\bar{\pi}$, introduced by the fundamental group π , has a right adjoint and consequently is right exact and preserves direct limits in the new categories. These properties enable us to present in section five a new version of Van Kampen theorem for joins of wild pseudo Peano continuum spaces. Finally, in section six, using the right exactness of $\bar{\pi}$, we study the behavior of the fundamental group on quotient spaces of wild pseudo Peano continua.

2 Definitions and Preliminaries

We are supposed that the reader is familiar with basic concepts and properties in group theory, topology, algebraic topology and category theory. In this section, we recall some essential facts from [1,2,3]. Firstly, given two categories \mathcal{C} and \mathcal{D} , a pair of functors (T_1, T_2) , where $T_1 : \mathcal{C} \rightarrow \mathcal{D}$ and $T_2 : \mathcal{D} \rightarrow \mathcal{C}$, is said to be *adjoint* if there exists a natural equivalence

$$\text{Hom}_{\mathcal{D}}(T_1 X, Y) \simeq \text{Hom}_{\mathcal{C}}(X, T_2 Y),$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. The functor T_2 is called the *right-adjoint functor* to T_1 and the pair (T_1, T_2) is called an *adjoint pair*.

It is well-known that every functor which has right-adjoint preserves direct limits [7], that is for any direct systems $\{X_i; \lambda_i^j, I\}$ indexed by a partially ordered set I , in the category \mathcal{C} , we have

$$T_1(\lim_{\rightarrow} X_i) = \lim_{\rightarrow} T_1(X_i).$$

In addition, we recall two important topological structures, join and weak join [4,7] of a family of pointed topological spaces $\{(X_i, x_i)\}$ indexed by a partially ordered set I . The *join* space $(X, x) = \bigvee_{i \in I} (X_i, x_i)$ of the X_i at the origin x is the quotient of the disjoint union of the X_i by the relation which identifies all the copies x_i of x . The *weak join* of these spaces, denoted by $(\tilde{X}, x) = \tilde{\bigvee}_{i \in I} (X_i, x_i)$, is defined as a different topology on the same underlying set, that of \tilde{X} is coarser. The topology of \tilde{X} is exactly the same everywhere except at the distinguished point x . A neighborhood of x in X is any set of the form $\bigvee_{i \in I} U_i$, where U_i is a neighborhood of x_i in X_i . A neighborhood of x in \tilde{X} has the same form but must also satisfy $U_i = X_i$ for almost all i .

Note that the above constructions, join and weak join spaces, can be considered as direct limit and inverse limit of special systems in the category

of pointed topological spaces, respectively, [8]. For the structure of direct limit and inverse limit in the category Top_* , we only recall that direct limit is a particular quotient of join space and inverse limit is a special subspace of product space. For further details we refer the reader to [8].

We also recall some topological definitions. A topological space X has *topological dimension* m , if every covering \mathcal{U} of X has a refinement \mathcal{U}' in which every point of X occurs in at most $m + 1$ sets in \mathcal{U}' , and m is the smallest such integer [5].

A locally path connected, path connected, compact metric space is called *Peano continuum* [1]. A space X is said to be *semilocally simply connected* (*semilocally 1-connected*) [6] at $x \in X$ if there exists an open neighborhood U_x of x so that the inclusion map $i : U_x \hookrightarrow X$ induces the trivial homomorphism $i_* : \pi(U_x, x) \hookrightarrow \pi(X, x)$, otherwise the point $x \in X$ is called a wild point at X .

We continue this section by pointing out some algebraic and topological concepts that have been studied in [1,3].

Let $\{G_i \mid i \in I\}$ be a family of groups, $\prod_{i \in X}^* G_i$ be the free product of the G_i for $X \subset I$, and $p_{XY} : \prod_{i \in Y}^* G_i \rightarrow \prod_{i \in X}^* G_i$ be the canonical homomorphism for $X \subset Y \subset I$. Then the unrestricted free product of the family is the inverse limit $\lim_{\leftarrow} \left(\prod_{i \in X}^* G_i, p_{XY} : X \subset Y \sqsubset I \right)$, where $Y \sqsubset I$ means that Y is the finite subset of I [3]. The free products of groups are defined by using words of finite length, and the infinitary version of the free products will be defined by using words of infinite length instead of finite one. Indeed our interest is concentrated on *free σ -product* of G_i ($i \in I$), denoted by $X_{i \in I}^\sigma G_i$, which is defined by using words of countable length and is a subgroup of the unrestricted free product [3].

Note that one of the most interesting theorems which we use in this paper asserts that the fundamental group of a weak join of a family of some

topological spaces is isomorphic to the free σ -product of their fundamental groups [3, Theorem A.1]. For instance, we point out the well-known *Hawaiian earring space*, is a weak join of countably many circles, whose fundamental group is isomorphic to the free σ -product $X_{n<\omega}^\sigma \mathbb{Z}_n$, where \mathbb{Z}_n , for $n < \omega$, is a copy of the infinite cyclic group \mathbb{Z} .

Now we mention one of the most essential structures, introduced by Conner and Eda in [1] and our results are mainly based on it. For an arbitrary group G , \mathcal{H}_G is defined to be the set of all subgroups of G which are homomorphic images of $X_{n<\omega}^\sigma \mathbb{Z}_n$. A finite subset F of \mathcal{H}_G is said to be *compatible* if there exists $H \in \mathcal{H}_G$ such that $\cup F \subseteq H$. A subfamily C of \mathcal{H}_G is said to be *compatible* if any finite subset of C is compatible. Let \mathcal{X}_G be the set of all maximal compatible subfamilies of \mathcal{H}_G which contain an uncountable subgroup. For subgroups H and H' of G , we denote $H \preceq H'$ if there exists $F \sqsubset G$ such that $H \leq \langle H' \cup F \rangle$.

Also, a topology on the set \mathcal{X}_G is introduced so that for any $Y \subseteq \mathcal{X}_G$, Y is closed if contains all its limit points. Note that a point x is called to be a limit point of Y if there exists a sequence $(x_n : n < \omega)$ of elements of Y satisfying the condition that for given uncountable $H_n \in x_n (n < \omega)$ there exists $H'_n \in x_n$ such that:

- $H_n \preceq H'_n$;
- for arbitrary $a_n \in H'_n (n < \omega)$ there exists $h : X_{n<\omega}^\sigma \mathbb{Z}_n \rightarrow G$ such that $h(\delta_n) = a_n$ for every $n < \omega$ and $\text{Im}(h) \in x$, where \mathbb{Z}_n is a copy of the infinite cyclic group \mathbb{Z} and δ_n it's generator for any $n < \omega$.

Note that one of the goal, studied by Conner and Eda [1], is to find a structure for the space \mathcal{X}_G with respect to the structure of G . For instance, we present some of these interesting results which are the most useful in our main results in this paper. In each point followed up, we refer to [1] for the proof and further details.

Theorem 2.1. (*[1, Theorem 2.2]*). *Let A be an abelian group. Then \mathcal{X}_A is*

an empty or one-point space. Specially $\mathcal{X}_{\mathbb{Z}} = \emptyset$.

Theorem 2.2. ([1, Theorem 5.1]). *Let X be a locally path connected, path connected, one-dimensional metric space and G be the fundamental group $\pi(X)$. Then \mathcal{X}_G is homeomorphic to X^w . Consequently, if X is wild, then \mathcal{X}_G is homeomorphic to X itself.*

Theorem 2.3. ([1, Corollary 5.2]). *Let X_n be a locally path connected, path connected, one-dimensional metric spaces such that $X_n^w \neq \emptyset$, for all $n < \omega$, and G be the fundamental group $\pi(\prod_{n < \omega} X_n) \simeq \prod_{n < \omega} \pi(X_n)$. Then \mathcal{X}_G is homeomorphic to $\prod_{n < \omega} X_n^w$. Consequently, if X_n is wild, for every $n < \omega$, then \mathcal{X}_G is homeomorphic to $\prod_{n < \omega} X_n$.*

Finally, a group S is said to be *n-slender* [1] if and only if for each homomorphism $h : X_{n < \omega} \mathbb{Z}_n \rightarrow S$, the set $\{n < \omega : h(\delta_n) \neq e\}$ is finite, where \mathbb{Z}_n is a copy of the infinite cyclic group \mathbb{Z} and δ_n it's generator for any $n < \omega$. The class \mathcal{S} consists of all the groups G such that for any non-trivial element $g \in G$, there exists an n -slender group S and a homomorphism $h : G \rightarrow S$ with $h(g) \neq e$. Of course, in group theory, we can call \mathcal{S} as the class of all *residually n-slender* groups.

Theorem 2.4. ([1, Corollary 4.7]). *Let A and B be groups in \mathcal{S} . Then \mathcal{X}_{A*B} is the topological sum of \mathcal{X}_A and \mathcal{X}_B .*

Remark 2.5. First we note that the class \mathcal{S} of groups is closed under free product. For a free product $G = \prod_{i \in I}^* G_i$ of groups G_i in \mathcal{S} and $g \in G$, we consider a sequence $g = g_{i_1} g_{i_2} \cdots g_{i_n}$ with $g_{i_j} \in G_{i_j}$ and their corresponding homomorphisms $h_i : G_i \rightarrow S_i$ with $h_i(g_i) \neq e$. The homomorphism $h : \prod_{i \in I}^* G_i \rightarrow \prod_{i \in J}^* S_i$ clearly satisfies $h(g) \neq e$, since

$$h(g_i) = \begin{cases} h_i(g_i) & \text{if } i \in J \\ e & \text{otherwise.} \end{cases}$$

Hence G belongs to \mathcal{S} . Now, using Theorem 2.4, we conclude that for any family $\{G_i\}_{i \in I}$ of groups in \mathcal{S} and any finite partition $\{J_1, \dots, J_n\}$ of the index set I , the space $\mathcal{X}_{\prod_{i \in I}^* G_i}$ is the topological sum of the spaces $\mathcal{X}_{*_{i \in J_1} G_i}, \dots, \mathcal{X}_{\prod_{i \in J_n}^* G_i}$.

Corollary 2.6. *For any free product $A = \prod_{i \in I}^* A_i$ of finitely many abelian groups which belong to the class \mathcal{S} , the space \mathcal{X}_G is empty or a finite space with the discrete topology. In particular, for any free group F of finite rank, the space \mathcal{X}_F is empty.*

3 Pseudo Peano Continuum Spaces

In foregoing, we are dealing with pseudo Peano continua introduced in section one.

Note 3.1. For any pseudo Peano continuum space X with $|X^w| \geq 2$, the fundamental group $\pi(X)$ is not abelian. Indeed, using Theorem 2.1, we know that for any abelian group A , \mathcal{X}_A (and so the wild subspace of a pseudo Peano continuum space whose fundamental group is isomorphic to A) is empty or one-point which implies the result.

Now, we are interested in analyzing the wild subspaces of joins and weak joins of pseudo Peano continuum spaces, which are important in our categorical arguments.

Lemma 3.2. *Let $\{(X_i, x_i)\}_{i \in I}$ be a finite family of pointed topological spaces which are pseudo Peano continuum. Then the join of these spaces, X say, will be also a pseudo Peano continuum space.*

Proof. Let $x_0 \in X$ be the common point of all X_i , so the path connectivity of all X_i implies the existence of paths from any point $x_0 \neq x \in X$, belonging

to a unique space X_{i_x} , to the base point x_0 . Hence the path connectivity of the space X is satisfied.

The locally path connectivity of the space X at any point $x_0 \neq x \in X$ is a direct result of the definition of join structure and locally path-connectivity of the X_i . In fact, for any neighborhood N_{x_0} of the special point $x_0 \in X$, for any $i \in I$, the neighborhood $(U_{x_0})_i = N_{x_0} \cap X_i$ should be open in the corresponding space X_i . Hence, by the locally path connectivity of any X_i , there exists the path connected neighborhood $(V_{x_0})_i \subseteq (U_{x_0})_i$. So the neighborhood $V = \cup_{i \in I} (V_{x_0})_i$ of x_0 as a subspace of N_{x_0} is our desirable neighborhood.

To show the metrizable of the space X , firstly we consider the corresponding metric d_i to X_i , for any $i \in I$. Now using these metrics, we define the following metric on the whole space X , so that for any two points $x, y \in X$ which belong to X_i and X_j , respectively, we have

$$d(x, y) = \begin{cases} d_i(x, y) & \text{if } i = j \\ d_i(x, x_0) + d_j(x_0, y) & \text{otherwise.} \end{cases}$$

We note that the well-definition of the metric d is concluded by the condition of being mutually disjoint for all X_i in the join space X . The others metric properties of d are easily deduced by consideration of d_i 's to be metrics.

Finally, the join space X is obviously one-dimensional. It is sufficient to note that the space X as a disjoint union of one-dimensional spaces is also one-dimensional. Indeed, for any covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X and any $i \in I$, we consider $\mathcal{U}_i = \{U_\lambda \cap X_i\}_{\lambda \in \Lambda}$ which is a cover for the one-dimensional space X_i and so it has a suitable refinement $\tilde{\mathcal{U}}_i$, say. Hence the union $\cup_{i \in I} \tilde{\mathcal{U}}_i$ is a cover of X satisfying the condition of being one-dimensional for the join space X . \square

Lemma 3.3. *Let $\{(X_i, x_i)\}_{i \in I}$ be a countable family of pointed topological*

spaces which are pseudo Peano continuum. Then the weak join of these spaces, (\tilde{X}, x_0) say, will be also a pseudo Peano continuum space.

Proof. Firstly, we recall that the topology of two spaces, join and weak join of all X_i , are exactly similar except at the base point x_0 . In order to prove the details, it is sufficient to investigate at the special point x_0 . Also we note that the path and the metric introduced in the proof of the lemma 3.2, with the similar argument, are those which show the path connectivity and the metrizable of the weak join space \tilde{X} .

For the locally path connectivity of \tilde{X} we consider the similar neighborhood of x_0 and note that there exists i_0 so that the neighborhood N_{x_0} contains the whole space X_j , for any $j \geq i_0$.

The one-dimensionality of \tilde{X} is clearly satisfied. For any covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of \tilde{X} , we consider the neighborhood U_{λ_0} of x_0 which, by the definition, contains all but a finite number of X_i 's, X_1, X_2, \dots, X_n say. Now we can consider \tilde{X} as the quotient space of the disjoint union $(U_{\lambda_0} \cap \tilde{X}) \cup X_1 \cup \dots \cup X_n$ that for any i , $1 \leq i \leq n$, $\mathcal{U}_i = \{U_\lambda \cap X_i\}_{\lambda \in \Lambda}$ is a cover for the one-dimensional space X_i and so it has a suitable refinement $\tilde{\mathcal{U}}_i$. Finally the union $(U_{\lambda_0} \cap \tilde{X}) \cup \tilde{\mathcal{U}}_1 \cup \dots \cup \tilde{\mathcal{U}}_n$ is a cover of \tilde{X} which obviously satisfied the condition of being one-dimensional for the weak join space \tilde{X} and this completes the proof. \square

At the end of this section, we summarize some important results of [2] in our new language which assert precisely that for any two wild pseudo Peano continuum spaces X and Y , the following conditions are equivalent:

- (i) X, Y are homeomorphic spaces.
- (ii) X, Y have the same homotopy type.
- (iii) The fundamental groups of spaces X, Y are isomorphic.

4 A Categorical Viewpoint

In this section, we are ready to mention a categorical viewpoint for the main concept of the paper i.e. pseudo Peano continuum space. Specially we try to get some functorial properties of fundamental groups which do not hold in general, but can be satisfied in the particular category which we are going to define.

The Category \mathbf{Top}^w ; which is a subcategory of \mathbf{Top}_* , consist of all spaces which are wild pseudo Peano continuum and of all morphisms between them.

The Category $\pi(\mathbf{Top}^w)$; which is a subcategory of the category **Group**, consists of all fundamental groups of wild pseudo Peano continuum spaces and of all homomorphisms induced by the map between them.

By considering a congruence relation \sim on the set of all morphisms in the category \mathbf{Top}^w such that $f \sim g$ if and only if the induced homomorphism $\pi(f) = f_*$ and $\pi(g) = g_*$ are equal in the category **Group**, we can construct the quotient category \mathbf{qTop}^w and it tends to define two functors $\bar{\pi}$ and \mathcal{X}_- as follows. As some famous objects of the category \mathbf{qTop}^w , we name the spaces Menger sponge, Sierpinski gasket, Sierpinski carpet.

Note that for our purpose, we need to consider some suitable subcategories of \mathbf{qTop}^w as follows:

By a suitable subcategory of \mathbf{qTop}^w , denoted by \mathbf{sqTop}^w , we mean all spaces in \mathbf{qTop}^w with a specific point which we correspond to each object with the following manner.

Let X be a wild pseudo Peano continuum space and $G = \pi(X)$ be its fundamental group. By Theorem 2.2 there exists a homeomorphism $\varphi : \mathcal{X}_G \rightarrow X$. Choose any fixed point $*_G$ in \mathcal{X}_G and consider $\varphi(*_G)$ as the specific corresponding point for the space X . Now the pointed space $(\mathcal{X}_G, *_G)$ is one of the objects of the suitable subcategory \mathbf{sqTop}^w . Note that for any pointed space (X, x_0) in \mathbf{qTop}^w , we can always choose a suitable fixed point $*_{\pi(X)}$ in

$\mathcal{X}_{\pi(X)}$ such that $\varphi(*_{\pi(X)}) = x_0$.

The functor $\bar{\pi}$; from a suitable subcategory of \mathbf{qTop}^w , \mathbf{sqTop}^w to $\pi(\mathbf{Top}^w)$ such that for any space X and any morphism $[f]$ in the category \mathbf{sqTop}^w , defines $\bar{\pi}(X) = \pi(X)$ and $\bar{\pi}([f]) = f_*$.

The functor \mathcal{X}_- ; from the category $\pi(\mathbf{Top}^w)$ to a category \mathbf{sqTop}^w such that for any group G and any morphism f_* in $\pi(\mathbf{Top}^w)$, defines $\mathcal{X}_-(G) = (\mathcal{X}_G, *_G)$ and $\mathcal{X}_-(f_*) = \mathcal{X}_{f_*} = [f]$.

Now with respect to the above notation, we state the following result.

Theorem 4.1. *The pair of functors $(\bar{\pi}, \mathcal{X}_-)$ is an adjoint pair.*

Proof. First, we note that the functors $\bar{\pi}$ and \mathcal{X}_- are inverse to each other. Indeed, using Theorem 2.2, for any $(X, x_0) \in \mathbf{sqTop}^w$, the space $\mathcal{X}_{\pi(X)}$ is homeomorphic to the wild subspace of X and so by definition of \mathbf{sqTop}^w , the pointed space $\mathcal{X}_- \circ \bar{\pi}(X) = (\mathcal{X}_{\pi(X)}, *_\pi(X))$ is equivalent to the space (X, x_0) itself. Conversely, for any $G \in \pi(\mathbf{Top}^w)$, the claim $\bar{\pi} \circ \mathcal{X}_-(G) \cong G$ is clearly satisfied.

Finally, for every space $(X, x_0) \in \mathbf{sqTop}^w$ and every group $G = \pi(\mathcal{X}_G, *_G) \in \pi(\mathbf{Top}^w)$, we define

$$\theta(= \theta_{X,G}) : \text{Hom}_{\pi(\mathbf{Top}^w)}(\bar{\pi}(X), G) \rightarrow \text{Hom}_{\mathbf{sqTop}^w}((X, x_0), (\mathcal{X}_G, *_G))$$

mapping f_* to $\mathcal{X}_{f_*} = [f]$, where f is a morphism in $\text{Hom}_{\mathbf{Top}^w}((X, x_0), (\mathcal{X}_G, *_G))$.

Firstly θ is well-defined because of the well-definition of \mathcal{X}_- , also it is one to one and onto due to the fact that $\bar{\pi}$ and \mathcal{X}_- are inverse to each other. Also the map θ , as a bijection, is natural in each variable; that is the following diagrams commute:

$$\begin{array}{ccc} \text{Hom}_{\pi(\mathbf{Top}^w)}(\bar{\pi}(X_1), G_1) & \xrightarrow{(\bar{\pi}(f))^*} & \text{Hom}_{\pi(\mathbf{Top}^w)}(\bar{\pi}(X_2), G_1) \\ \downarrow \theta & & \downarrow \theta \\ \text{Hom}_{\mathbf{sqTop}^w}(X_1, \mathcal{X}_{G_1}) & \xrightarrow{(f)^*} & \text{Hom}_{\mathbf{sqTop}^w}(X_2, \mathcal{X}_{G_1}) \end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Hom}_{\pi(\mathbf{Top}^w)}(\bar{\pi}(X_1), G_1) & \xrightarrow{(g_*)^*} & \mathrm{Hom}_{\pi(\mathbf{Top}^w)}(\bar{\pi}(X_1), G_2) \\
\downarrow \theta & & \downarrow \theta \\
\mathrm{Hom}_{\mathbf{sqTop}^w}(X_1, \mathcal{X}_{G_1}) & \xrightarrow{(\mathcal{X}_{g_*})^*} & \mathrm{Hom}_{\mathbf{sqTop}^w}(X_1, \mathcal{X}_{G_2})
\end{array}$$

for all $f : X_2 \rightarrow X_1$ in \mathbf{sqTop}^w and $g_* : G_1 \rightarrow G_2$ in $\pi(\mathbf{Top}^w)$. This note ends the proof. \square

The above theorem may be the most essential fact in this paper so that the existence of the right adjoint for a functor can imply some known manner such as preserving direct limits, if any, and keeping the right exactness of an exact sequence, in a certain sense. We end with the following result.

Theorem 4.2. *For every direct system $\{(X_i, x_i), i \in I\}$ of spaces in a subcategory \mathbf{sqTop}^w whose direct limit belongs to the category, we have the isomorphism $\pi(\varinjlim X_i) \cong \varinjlim \pi(X_i)$.*

Proof. The functor $\bar{\pi}$ which has a right adjoint \mathcal{X}_- , should preserve the direct limit [7], that is the above isomorphism and so the result holds. \square

5 A Theorem of Van Kampen Type for Join Spaces

In this section, we are in a position to present one of the main result of the paper which is a theorem of Van Kampen type theorem for wild pseudo Peano continuum spaces. Note that the well-known Van Kampen Theorem asserts that the fundamental group of a join of first countable semilocally simply connected spaces is isomorphic to the free product of their fundamental groups [4]. However, the conditions presented in the following theorem are completely different from (even though opposite to) the conditions in Van Kampen Theorem.

Lemma 5.1. *The join of any finite family of spaces in \mathbf{sqTop}^w is also in this category.*

Proof. Let $\{X_i\}_{i \in I}$ be a finite family of spaces in \mathbf{sqTop}^w and $(X, x_0) = \bigvee (X_i, x_i)$ be the join space of this family. First, we recall the definition of \mathbf{sqTop}^w and the note after it, also Lemma 3.2 which asserts that the join space of pseudo Peano continuum spaces is also pseudo Peano continuum. So to complete the proof, it is sufficient to prove the space X is also wild. By the contrary, suppose there exists a point $x \in X$ and a neighborhood U_x of it so that the homomorphism $\pi(j) : \pi(U_x) \rightarrow \pi(X)$, induced by inclusion, is trivial. Now if we composite the inclusion map $l_i : U_x \cap X_i \rightarrow X_i$ and the collapsing map $p_i : X \rightarrow X_i$, we obtained the inclusion $p_i \circ j \circ l_i : U_x \cap X_i \rightarrow X_i$ which induces the trivial homomorphism $\pi(p_i \circ j \circ l_i) = \pi(p_i)\pi(j)\pi(l_i) : \pi(U_x \cap X_i) \rightarrow \pi(X_i)$ which contradicts to supposition of X_i to be wild space. \square

Theorem 5.2. *(fundamental groups of join spaces in \mathbf{Top}^w). The fundamental group of the join of a finite family of wild pseudo Peano continuum spaces is isomorphic to the coproduct of their fundamental groups in $\pi(\mathbf{Top}^w)$.*

Proof. Firstly, we recall that the join space $(X, x) = \bigvee (X_i, x_i)$ as coproduct of the finite family $\{X_i, i \in I\}$ is the direct limit of a special direct system explained in [7,8]. Now, by considering a suitable subcategory \mathbf{sqTop}^w contains all the pointed space (X_i, x_i) and using Lemma 5.1 and Theorem 4.2, the fundamental group of $(X, x) = \bigvee (X_i, x_i)$ must be the direct limit of the induced corresponding direct system, which plays the role of the coproduct of $\{\pi(X_i)\}_{i \in I}$ in the category $\pi(\mathbf{Top}^w)$. Since $\bigvee (X_i, x_i)$ is a wild pseudo Peano continuum space and $\pi(\bigvee (X_i, x_i)) \cong \varinjlim \pi(X_i, x_i)$, the above coproduct in the category $\pi(\mathbf{Top}^w)$ does exist. \square

As we see, the results of this section are deduced essentially by using the functorial property of the fundamental group on the special category which

we constructed. Now attending to the functor \mathcal{X}_- as a right adjoint to $\bar{\pi}$, we will get a result which is indeed another proof for Theorem 2.3 [1, Corollary 5.2]. Details are offered in the next section.

6 Fundamental Groups of Quotient Spaces

In this final section we are going to find a relation between fundamental groups of a quotient space X/A , the space X and its subspace A , where X/A , X , and A belong to the category \mathbf{Top}^w . In order to do this, we need to consider cokernels as direct limits in \mathbf{sqTop}_0^w , but this category has no zero object. In order to compensate this default we consider the following two categories:

The Category \mathbf{sqTop}_0^w ; is obtained by \mathbf{sqTop}^w by adding the point $\{*\}$ and all morphisms, the constant map $0^X : (X, x_0) \rightarrow \{*\}$, and the map $0_Y : \{*\} \rightarrow (Y, y_0)$ taking $*$ to the point y_0 , to the category \mathbf{sqTop}^w .

The Category $\pi(\mathbf{Top}_0^w)$; is obtained by $\pi(\mathbf{Top}_0^w)$ by adding the trivial group $\{e\}$ and all morphisms, the trivial homomorphism $0^{\pi(X)} : \pi(X, x_0) \rightarrow \{e\}$ and the homomorphism $0_{\pi(Y)} : \{e\} \rightarrow \pi(Y, y_0)$ mapping e to the identity e , to the category $\pi(\mathbf{Top}^w)$.

Moreover, we should consider two functors $\bar{\pi}_0 : \mathbf{sqTop}_0^w \rightarrow \pi(\mathbf{Top}_0^w)$ and $\mathcal{X}_{0-} : \pi(\mathbf{Top}_0^w) \rightarrow \mathbf{sqTop}_0^w$ which are natural extensions of $\bar{\pi}$ and \mathcal{X}_- , and correspond added objects $\{*\}$ and $\{e\}$ to each other, such as:

$$\bar{\pi}_0 : \{*\} \longmapsto \{e\} \quad \text{and} \quad \mathcal{X}_{0-} : \{e\} \longmapsto \{*\}.$$

Similar to Theorem 5.1, one can easily verify that $(\bar{\pi}_0, \mathcal{X}_{0-})$ is an adjoint pair, and hence $\bar{\pi}_0$ preserves direct limits in \mathbf{sqTop}_0^w , and \mathcal{X}_{0-} preserves inverse limits in $\pi(\mathbf{Top}_0^w)$.

Now, consider a pointed space (X, x_0) and a subspace A containing x_0 such that (X, x_0) , (A, x_0) , and $(X/A, *)$ belong to \mathbf{sqTop}_0^w . Note that we

can choose a suitable fixed point $*_{\pi(X)}$ in $\mathcal{X}_{\pi(X)}$ and $*_{\pi(A)}$ in $\mathcal{X}_{\pi(A)}$ such that $\varphi(*_{\pi(X)}) = x_0$ and $\psi(*_{\pi(A)}) = x_0$, where $\psi : \mathcal{X}_{\pi(A)} \rightarrow A$ is the homeomorphism.

We can consider the quotient space $(X/A, *)$ as a pushout of the following diagram:

$$\begin{array}{ccc} (A, x_0) & \xrightarrow{j} & (X, x_0) \\ \downarrow 0^A & & \downarrow p \\ \{*\} & \xrightarrow{nat} & (X/A, *), \end{array}$$

where j and p are inclusion and quotient maps, respectively. The map nat also naturally corresponds $*$ to the point $*$.

Therefore the pointed space $(X/A, *)$ can be considered as a direct limit in \mathbf{sqTop}_0^w , and hence $\pi(X/A, *)$ is the pushout of the following diagram:

$$\begin{array}{ccc} \pi(A, x_0) & \xrightarrow{j_*} & \pi(X, x_0) \\ \downarrow 0^{\pi(A)} & & \downarrow p_* \\ \{e\} & \xrightarrow{nat_*} & \pi(X/A, *), \end{array}$$

where j_* and p_* are the induced map $\pi(j)$ and $\pi(p)$, respectively. The homomorphism nat_* is also mapping e to the identity of $\pi(X/A, *)$. Also, we know that $\pi(X, x_0)/j_*(\pi(A, x_0))^{\pi(X, x_0)}$ is the pushout of the above diagram in the category of groups. Hence there exists a unique homomorphism

$$\psi : \frac{\pi(X, x_0)}{j_*(\pi(A, x_0))^{\pi(X, x_0)}} \rightarrow \pi(X/A, *)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi(X, x_0) & \xrightarrow{p_*} & \pi(X/A, *) \\ \downarrow nat & & \nearrow \psi \\ \pi(X, x_0)/j_*(\pi(A, x_0))^{\pi(X, x_0)} & & \end{array}$$

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